

International Journal of Innovative Research in Science, Engineering and Technology

(A High Impact Factor & UGC Approved Journal)

Website: www.ijirset.com

Vol. 6, Issue 8, August 2017

Newton's Law of Cooling - Elzaki Transform

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ABSTRACT: In this paper, we solved the problems occurred in first order ordinary linear differential equations based on Newton's Law of cooling by using Elzaki Transform.

KEYWORDS: Elzaki Transform, Newton's Law of cooling, Differential equations.

I. INTRODUCTION

In science and engineering Differential equations plays an important role. In order to solve the differential equations, the integral transforms were extensively used. The importance of an Integral Transforms is that they provide powerful operational methods for solving initial value problems. Mostly we use Laplace Transform technique for solving differential equations. Elzaki Transform is the new integral transform which is stronger than Sumudu and Laplace Transforms.

Elzaki Transform was introduced by Tarig Elzaki [1] in 2011. We apply this new transform technique for solving the problems on law of natural growth and decay.

Elzaki Transform defined for function of exponential order, we consider function on the set A defined by

$$A = \left\{ f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}$$

For a given function in the set A the constant

M must be a finite number, k_1 and k_2 may be finite or infinite. The Elzaki transform is defined by

$$E[f(t)] = v \int_0^{\infty} f(t) e^{\frac{-t}{v}} dt = T(V), t \geq 0, k_1 \leq v \leq k_2.$$

The sufficient conditions for the existence of the Elzaki transform are that $f(t)$ for $t \geq 0$ be piecewise continuous and of the exponential order otherwise Elzaki transform may (or) may not exist.

II. ELZAKI TRANSFORM OF SOME FUNCTIONS AND FOR FIRST DERIVATE

Here we consider some standard functions and the first derivative which are mostly occurred in the problems on Newton's law cooling; their Elzaki transforms are given below.

$$(i) \quad \text{If } f(t)=1, \quad \text{Now } E\{f(t)\} = v \int_0^{\infty} e^{\frac{-t}{v}} f(t) dt = v \int_0^{\infty} e^{\frac{-t}{v}} dt = v \left(\frac{e^{\frac{-t}{v}}}{\frac{-1}{v}} \right)_0^{\infty} = v^2 \quad \therefore E\{1\} = v^2$$

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$$(ii) \quad \text{If } f(t)=t, \text{ then } E\{f(t)\} = v \int_0^{\infty} e^{-\frac{t}{v}} f(t) dt \Rightarrow E\{t\} = v \int_0^{\infty} t e^{-\frac{t}{v}} dt$$

$$= v \left\{ t \left(\frac{e^{-\frac{t}{v}}}{-\frac{1}{v}} \right) - (1) \left(\frac{e^{-\frac{t}{v}}}{\frac{1}{v^2}} \right) \right\}_0^{\infty} = v(v^2) = v^3 \quad \therefore E(t) = v^3$$

$$(iii) \quad \text{Similarly, we get } E(t^n) = n! v^{n+2}$$

$$(iv) \quad E(e^{at}) = v \int_0^{\infty} e^{at} e^{-\frac{t}{v}} dt = \frac{v^2}{1-av}$$

Theorem: Let $f(t)$ be the given function and $E\{f(t)\} = T(v)$ then $E\{f^1(t)\} = \frac{T(v)}{v} - vf(0)$

Proof: Given $E\{f(t)\} = T(v)$, By Definition of Elzaki Transform $E\{f(t)\} = v \int_0^{\infty} f(t) e^{-\frac{t}{v}} dt$

$$\text{Now } E\{f^1(t)\} = v \int_0^{\infty} f^1(t) e^{-\frac{t}{v}} dt$$

$$= v \left\{ \left(e^{-\frac{t}{v}} f(t) \right)_0^{\infty} - \int_0^{\infty} e^{-\frac{t}{v}} \left(\frac{-1}{v} \right) f(t) dt \right\} = v \left\{ -f(0) + \frac{1}{v} \int_0^{\infty} f(t) e^{-\frac{t}{v}} dt \right\}$$

$$= -vf(0) + \int_0^{\infty} f(t) e^{-\frac{t}{v}} dt = -vf(0) + \frac{T(v)}{v} \quad \therefore E\{f^1(t)\} = \frac{T(v)}{v} - vf(0)$$

III. NEWTON'S LAW OF COOLING

Newton's law of cooling states that the rate of change of the temperature of a body is proportional to the difference of the temperature of the body and that of its surrounding medium.

Let θ be the temperature of the body at any time t , θ_0 be the temperature of its surrounding medium. Then by Newton's law of cooling we have

$$\frac{d\theta}{dt} \propto (\theta - \theta_0) \Rightarrow \frac{d\theta}{dt} = -k(\theta - \theta_0), \text{ where } k \text{ is proportionality constant } (k > 0)$$

IV. ELZAKI TRANSFORM ON RELATED PROBLEMS

Example-1: The temperature of a body drops from 80°C to 60°C . In 20 minutes when the surrounding air is at 40°C . Then what will be its temperature after 40 minutes and when will be the temperature 55°C ?

Solution: Given that, the surrounding temperature $\theta_0 = 40^{\circ}\text{C}$, initial temperature i.e. at time $t = 0$, $\theta(0) = 80^{\circ}\text{C}$ and after time $t = 20 \text{ min}$, $\theta(20) = 60^{\circ}\text{C}$.

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From Newton's Law of cooling, we have

$$\frac{d\theta}{dt} \propto (\theta - \theta_0) \Rightarrow \frac{d\theta}{dt} = -k(\theta - \theta_0) \Rightarrow \frac{d\theta}{dt} = -k\theta + k\theta_0 \Rightarrow \frac{d\theta}{dt} + k\theta = k\theta_0 \Rightarrow \theta'(t) + k\theta = 40k$$

Taking Elzaki Transform both sides, we get

$$\begin{aligned} E\{\theta'(t) + k\theta\} &= E\{40k\} \Rightarrow E\{\theta'(t)\} + kE\{\theta\} = 40kE\{1\} \\ \Rightarrow \left[\frac{\bar{\theta}(v)}{v} - v\theta(0) \right] + k\bar{\theta}(v) &= 40kv^2 \quad \text{-----(1)} \end{aligned}$$

Since at time t=0, given that $\theta(0) = 80^\circ c$, substituting this in equation (1), we have

$$\begin{aligned} \left[\frac{\bar{\theta}(v)}{v} - 80v \right] + k\bar{\theta}(v) &= 40kv^2 \\ \Rightarrow \left(\frac{1}{v} + k \right) \bar{\theta}(v) &= 80v + 40kv^2 \\ \Rightarrow (1 + kv) \bar{\theta}(v) &= 80v^2 + 40kv^3 \\ \Rightarrow \bar{\theta}(v) &= \frac{80v^2}{(1 + kv)} + \frac{40kv^3}{(1 + kv)} \quad \text{-----(2)} \end{aligned}$$

Applying inverse Elzaki Transform on both side of equation (2), then we get

$$\begin{aligned} \Rightarrow E^{-1}\{\bar{\theta}(v)\} &= E^{-1}\left\{ \frac{80v^2}{(1 + kv)} + \frac{40kv^3}{(1 + kv)} \right\} \\ \Rightarrow \theta(t) &= 80 E^{-1}\left\{ \frac{v^2}{(1 + kv)} \right\} + 40k E^{-1}\left\{ \frac{v^3}{(1 + kv)} \right\} \end{aligned}$$

Since $E^{-1}\left\{ \frac{v^2}{(1 + kv)} \right\} = e^{-kt}$ and $\frac{v^3}{(1 + kv)} = \frac{1}{k}v^2 - \frac{1}{k} \cdot \frac{v^2}{(1 + kv)}$, so we get

$$\theta(t) = 80e^{-kt} + 40k E^{-1}\left[\frac{1}{k}v^2 - \frac{1}{k} \cdot \frac{v^2}{(1 + kv)} \right] = 80e^{-kt} + 40k \left[\frac{1}{k} - \frac{1}{k} e^{-kt} \right] = 80e^{-kt} + 40 - 40e^{-kt}$$

Thus $\theta(t) = 40 + 40e^{-kt}$ -----(3)

Using the condition at t=20min, $\theta(20)=60^\circ c$, from (3) we have

$$60 = 40 + 40e^{-20k} \Rightarrow 20 = 40e^{-20k} \Rightarrow \frac{1}{2} = e^{-20k} \Rightarrow e^{-k} = \left(\frac{1}{2} \right)^{1/20} \quad \text{-----(4)}$$

To find the temperature after time t=40min, from (3) we have $\theta(t) = 40 + 40e^{-40k}$

$$\theta(t) = 40 + 40(e^{-k})^{40} = 40 + 40\left(\frac{1}{2} \right)^{20} = 40 + 40\left(\frac{1}{2} \right)^2 = 50^\circ c.$$

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Now to find the time required to become the temperature of the body is 55°C , again from (3), we get $55 = 40 + 40e^{-kt} \Rightarrow 15 = 40(e^{-k})^t \Rightarrow \frac{15}{40} = (e^{-k})^t$ _____(5)

Substituting (4) in (5), we get $\frac{15}{40} = \left(\frac{1}{2}\right)^{t/20}$. Taking the logarithms on both side,

$$\ln\left(\frac{15}{40}\right) = \frac{t}{20} \ln\left(\frac{1}{2}\right) \Rightarrow t = \frac{20 \ln\left(\frac{15}{40}\right)}{\ln\left(\frac{1}{2}\right)} \text{ min} = 28.3 \text{ min utes}$$

Example-2: (Estimation of time of murder) The body of murder victim was discovered at 11:00 PM the doctor took the temperature of body at 11:30PM which was 94.6°F . He again took the temperature after one hour when it showed 93.4°F , and noticed that the temperature of the room was 70°F . Estimate the time of death (Normal temperature of human body= 98.6°F).

Solution: Given that, the room temperature is $\theta_0=70^{\circ}\text{F}$. The recorded temperature at 11:30 PM is considered as a initial temperature i.e. At time $t=0$, $\theta(0)=94.6^{\circ}\text{F}$.

From Newton's Law of cooling, we have $\frac{d\theta}{dt} + k\theta = k\theta_0 \Rightarrow \theta'(t) + k\theta(t) = 70k$

Taking Elzaki Transform on both sides, we get

$$E\{\theta'(t) + k\theta(t)\} = 70k E\{1\} \Rightarrow E\{\theta'(t)\} + k E\{\theta(t)\} = 70k E\{1\}$$

$$\Rightarrow \left[\frac{\bar{\theta}(v)}{v} - v\bar{\theta}(0) \right] + k\bar{\theta}(v) = 70k v^2$$
 _____(6)

Since at time $t=0$, given that $\theta(0) = 94.6^{\circ}\text{F}$, substituting this in equation (6), we have

$$\left(\frac{1}{v} + k\right) \bar{\theta}(v) = 70k v^2 + 94.6 v \Rightarrow (1 + kv)\bar{\theta}(v) = 70k v^3 + 94.6 v^2$$

$$\Rightarrow \bar{\theta}(v) = 70k \frac{v^3}{(1 + kv)} + 94.6 \frac{v^2}{1 + kv}$$
 _____(7)

Applying inverse Elzaki Transform on both side of equation (7), then we get

$$E^{-1}\{\bar{\theta}(v)\} = E^{-1}\left\{ \frac{70k v^3}{(1 + kv)} + \frac{94.6 v^2}{(1 + kv)} \right\}$$

$$\Rightarrow \theta(t) = 70k E^{-1}\left\{ \frac{v^3}{(1 + kv)} \right\} + 94.6 E^{-1}\left\{ \frac{v^2}{(1 + kv)} \right\}$$

Since $E^{-1}\left\{ \frac{v^2}{(1 + kv)} \right\} = e^{-kt}$ and $\frac{v^3}{(1 + kv)} = \frac{1}{k} v^2 - \frac{1}{k} \cdot \frac{v^2}{(1 + kv)}$, so we get

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$$\begin{aligned} \theta(t) &= 94.6 e^{-kt} + 70k E^{-1} \left[\frac{1}{k} v^2 - \frac{1}{k} \cdot \frac{v^2}{(1+kv)} \right] = 94.6 e^{-kt} + 70k \left[\frac{1}{k} - \frac{1}{k} e^{-kt} \right] \\ &= 70k \left[\frac{1}{k} - \frac{1}{k} e^{-kt} \right] + 94.6 e^{-kt} = 70 - 70e^{-kt} + 94.6 e^{-kt} \quad \therefore \theta(t) = 70 + 24.6 e^{-kt} \quad \text{---(8)} \end{aligned}$$

Using the condition after one hour i.e. at $t=60\text{min}$, $\theta(60) = 93.4^\circ\text{F}$, from (8) we have

$$93.4 = 70 + 24.6 e^{-60k} \Rightarrow 23.4 = 24.6 (e^{-k})^{60} \Rightarrow \left(\frac{23.4}{24.6} \right)^{1/60} = e^{-k} \quad \text{---(9)}$$

But the human body temperature before the death is 98.6°F , from (8)

$$98.6 = 70 + (24.6) e^{-kt} \Rightarrow 28.6 = (24.6) (e^{-k})^t \quad \text{---(10)}$$

Substituting (9) in (10), we get $\frac{28.6}{24.6} = \left(\frac{23.4}{24.6} \right)^{t/60}$. Taking logarithm on both sides, we get

$$\ln \frac{28.6}{24.6} = \frac{t}{60} \ln \left(\frac{23.4}{24.6} \right) \Rightarrow t = 60 \frac{\log \frac{28.6}{24.6}}{\log \left(\frac{23.4}{24.6} \right)} = -180.75 \text{ min} \approx 3 \text{ hours}$$

Therefore estimated death time = 11:30PM - 3:00 hours = 8:30PM. (Approximately)

Example-3: When a cake is removed from an oven its temperature is measured at 300°F . 3 minutes later its temperature is 200°F . How long will it take to cool off to a room temperature of 70°F .

Solution: Given that, the room temperature is $\theta_0 = 70^\circ\text{F}$, Initial temperature i.e. at time $t=0$, $\theta(0) = 300^\circ\text{F}$.

From Newton's Law of cooling, we have $\frac{d\theta}{dt} + k\theta = k\theta_0 \Rightarrow \theta'(t) + k\theta(t) = 70k$

Taking Elzaki Transform on both sides, we get

$$\begin{aligned} E \{ \theta'(t) + k\theta(t) \} &= 70k E \{ 1 \} \\ \Rightarrow E \{ \theta'(t) \} + k E \{ \theta(t) \} &= 70k E \{ 1 \} \\ \Rightarrow \left[\frac{\bar{\theta}(v)}{v} - v\bar{\theta}(0) \right] + k\bar{\theta}(v) &= 70k v^2 \quad \text{---(11)} \end{aligned}$$

Since at time $t=0$, given that $\theta(0) = 300^\circ\text{F}$, substituting this in equation (11), we have

$$\begin{aligned} \left(\frac{1}{v} + k \right) \bar{\theta}(v) &= 70k v^2 + 300v \\ \Rightarrow (1 + kv) \bar{\theta}(v) &= 70k v^3 + 300v^2 \\ \Rightarrow \bar{\theta}(v) &= 70k \frac{v^3}{(1 + kv)} + 300 \frac{v^2}{1 + kv} \end{aligned}$$

Applying inverse Elzaki Transform on both side of equation (7), then we get

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$$\Rightarrow E^{-1} \{ \bar{\theta}(v) \} = E^{-1} \left\{ \frac{70kv^3}{(1+kv)} + \frac{300v^2}{(1+kv)} \right\}$$

$$\Rightarrow \theta(t) = 70kE^{-1} \left\{ \frac{v^3}{(1+kv)} \right\} + 300E^{-1} \left\{ \frac{v^2}{(1+kv)} \right\} = 70k \left[\frac{1}{k} - \frac{1}{k} e^{-kt} \right] + 300e^{-kt} = 70 - 70e^{-kt} + 300e^{-kt}$$

$$\therefore \theta(t) = 70 + 230e^{-kt} \text{ (12)}$$

Using second condition, at time t= 3minutes given that $\theta(3) = 200^{\circ}\text{F}$, from (12) we have

$$200 = 70 + 230e^{-3k} \Rightarrow 130 = 230e^{-3k} \Rightarrow e^{-3k} = \frac{13}{23} \Rightarrow e^{-k} = \left(\frac{13}{23} \right)^{1/3} \text{ (13)}$$

However Equation (12) gives no finite solution to $\theta(t)=70^{\circ}\text{F}$, since $\lim_{t \rightarrow \infty} \theta(t) = 70$.

We expect the cake will at room temperature after a long period of time .This period is given in the tabular form.

$\theta(t)$ in $^{\circ}\text{F}$	Time (min)
75	20.1
74	21.3
73	22.8
72	24.9
71	28.6
70.5	32.3

From above we can conclude that the cake will approximately be at room temperature in about half an hour.

V. CONCLUSION

We can apply Elzaki Transform to solve the problems related to Newton’s Law of Cooling principle.

REFERENCES

- [1] Tarig.M.Elzaki, The New Integral Transform“Elzaki Transform“Global Journal of Pure and Applied Mathematics, Vol.7, Number-1, pp.57-64, 2011.
- [2] Alan Jeffrey, “Advanced Engineering Mathematics”, Academic Press, 2002.
- [3] Hasen Eltayeh and Adem Kilicman, “On Some Application of a New Integral Transform”, International Journal of Math Analysis, Vol.4, No.3, pp.123-132, 2010.
- [4] J.Zhang and A Sumudu, “Based Algorithm for Solving Differential Equations”, Comp.Sci.J. Moldova Vol.15, No.3, pp.303-313, 2007.
- [5] G Kwatugala, Sumudu Transform-“An Integral Transform to Solve Differential Equations Control Engineering Problems”, International J.Math.Ed.Sci.Tech, 24, pp.35-43, 1993.